Sensor Placement in Structural Control

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Issues associated with the placement of sensors in structural control and reliability systems are analyzed. The sensor output is used to extract modal coordinates. Optimization procedures are used to determine the most desirable locations of the operational sensors. It is shown that modal filtering by means of spline functions is more desirable for modal coordinate extraction because of stability, robustness, and convergence considerations. Selection of the number and location of the backup sensors is considered. It is observed that an even distribution of the operational and backup sensors yields satisfactory results. However, an uneven placement of sensors diminishes the quality of the modal coordinate extraction.

I. Introduction

An important issue in the control design for elastic systems is the determination of the number and location of the control system components, namely actuators and sensors. Because sensors are used not only for calculating the feedback control gains, but also for monitoring the reliability of the control system and of its components and for monitoring the accuracy of the mathematical model, selection of the number and location of the sensors is critical. The determination of the optimal number and placement of backup components, which are to be brought on-line in the event of failure of one of the operational sensors, is an additional issue in the control design.

This paper is concerned with the optimization of the number and location of sensors that are necessary for the monitoring, identification, and vibration suppression of a flexible structure. The problem of determining the optimal number and locations of backup sensors is considered as well. It is assumed that the control and monitoring of the reliability of the system model and of the control components is based on modal information.

Recent work in determining the number and location of control system components in distributed-parameter systems include Refs. 1-9, where a variety of criteria have been considered. References 1 and 2 use an optimal control cost minimization criterion. Minimization of the control energy is sought in Ref. 3, and minimization of the expected value of a performance index is sought in Ref. 4. Reliability issues are used in Refs. 5 and 6. In Ref. 5, mission life is considered, and the integral of the square of the modal amplitudes is used as a performance measure. In Ref. 6, possible actuator failures and the controllability of the system after failure has occurred is considered. The degree of controllability is also used as a criterion in Ref. 7. Failures of actuators and sensors are investigated in Refs. 8 and 9, respectively, where criteria are proposed to locate the actuators and sensors in a way that facilitates the failure-detection process. The results of Refs. 3, 8, and 9 indicate that an even distribution of the control components generally yields satisfactory results. General reliability and replacement of control components is considered in Refs. 10 and 11.

In this paper, we assume that the output of the sensors is used to extract modal coordinates. To this end, two approaches exist: modal filtering and Luenberger observers.[12,13] We consider here modal filters and associated implementation issues, namely the types of interpolation functions and the number of sensors to use for satisfactory performance. Previously, two types of interpolation functions were considered: finite-element type and Rayleigh-Ritz type.[12,13] We introduce here spline functions and compare the three types of interpolation functions. The comparison is carried out qualitatively for accuracy and convergence of the modal extraction and to identify the number of sensors required for satisfactory performance. It is shown that in general spline functions are more desirable because they provide a better interpolation of the deformation profile and they are less sensitive to unevenness in the sensor locations. They also have better convergence characteristics.

Next, optimization of the location of the operational and backup sensors is considered, where an objective function measuring the quality of the modal coordinate extraction is minimized. Note that measuring the accuracy of the modal coordinate extraction is an indirect measure of observability. Numerical examples are presented to illustrate the optimization procedures and as a basis to compare modal filters for different types of interpolation functions.

II. Equations of Motion and Modal Coordinate Extraction

Consider the equations of motion of a structure, expressed in the form

$$Lu(x,t) + m(x)\ddot{u}(x,t) = f(x,t) \tag{1}$$

where $u(x,t)$ is the deformation at x at time t, L is the stiffness operator, $m(x)$ is the mass, and $f(x,t)$ is the external excitation, including controls. Using standard methods of analysis, the modal equations of motion can be obtained as

$$\ddot{u}_r(t) + \omega_r^2 u_r(t) = f_r(t), \qquad r = 1,2,\ldots \tag{2}$$

in which $u_r(t)$ are modal coordinates, $f_r(t)$ are modal forces and ω_r are the natural frequencies. The modal coordinates and modal forces are related to $u(x,t)$ and $f(x,t)$ as

$$u(x,t) = \sum_{r=1}^{\infty} \phi_r(x) u_r(t) \tag{3a}$$

Received Aug. 4, 1988; presented as Paper 88-4056 at the AIAA Guidance, Navigation, and Control Conference, Mineapolis, MN, Aug. 15-17, 1988; revision received Nov. 30, 1988. Copyright © 1988 by Haim Baruh. Published by the American Institute of Aeronautics and Astronautics, Inc., with permission.

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$$u_r(t) = (u(x,t),\ m(x)\phi_r(x)) \tag{3b}$$

$$f(x,t) = \sum_{r=1}^{\infty} m(x)\phi_r(x)f_r(t) \tag{3c}$$

$$f_r(t) = (f(x,t),\ \phi_r(x)) \tag{3d}$$

where $(a,b) = \int ab\ \mathrm{d}x$, and $\phi_r(x)$ are the eigenfunctions, which satisfy $(\phi_r(x),\ m(x)\phi_s(x)) = \delta_{rs}$, $(\phi_r(x),\ L\phi_s(x)) = \lambda_r\delta_{rs} = \omega_r^2\delta_{rs}$ $(r,s = 1,2,\ldots)$.

In many applications, one needs to extract the modal coordinates from the system output. One such case is modal control,[12,14] where the feedback gains are a function of the modal coordinates. Another case is modal parameter identification,[15-17] where the determination of the system governing parameters is carried out by examining the modal output. Another application is component failure detection,[5,8,9] where the modal output is used to determine the reliability of a control system and to identify any possible sensor or actuator malfunctions. To extract the modal coordinates from the system output we consider in this paper modal filtering.[12,13]

In modal filtering, use is made of the expansion theorem [Eqs. (3a) and (3b)] to extract the modal coordinates from the system output. This requires spatially continuously distributed measurements, which are not within the state of the art. Implementation of modal filters using discrete measurements requires generation of an approximation distributed profile by interpolation or extrapolation of the sensors measurements as

$$q(x,t) = \sum_{j=1}^{\ell} u(x_j,t)G(x,x_j) \tag{4}$$

where $q(x,t)$ is the approximated profile, x_j denotes the location of the sensors $(j = 1,2,\ldots,\ \ell)$, and $G(x,x_j)$ are interpolation or extrapolation functions, in which the time and space dependency are separate. Note that such an approach can easily be extended to cases where the measurements are piecewise-continuous. Also, the design and implementation of modal filters are independent of the number and location of the system actuators.

Introducing Eq. (4) into the expansion theorem, we obtain expressions for the extracted modal coordinates $q_r(t)$, defined as

$$q_r(t) = (q(x,t),\ m(x)\phi_r(x)) = \sum_{j=1}^{\ell} g_{rj}u(x_j,t), \qquad r = 1,2,\ldots \tag{5}$$

where

$$g_{rj} = (G(x,x_j),\ m(x)\phi_r(x))\ (j = 1,2,\ldots,\ \ell;\ r = 1,2,\ldots)$$

The accuracy of modal filtering depends on the number and location of the sensors as well as on the type of interpolation or extrapolation functions used. In the existing literature, finite-element and Rayleigh-Ritz-type interpolation functions have been used to implement modal filters.[12,13] Here, we summarize these approaches and demonstrate the use of splines as suitable interpolation functions. We denote the number of modes of interest by m (referred to as monitored modes) and partition the modal coordinates into their monitored and residual components as $\boldsymbol{u}(t) = [u_1(t)\ u_2(t)\cdots]^T = [\boldsymbol{u}_M^T(t)\,|\,\boldsymbol{u}_R^T(t)]^T$. We also introduce the vector $q_M(t)$, where $q_M(t) = [q_1(t)\ q_2(t)\cdots q_m(t)]^T$ and express the sensor measurements as $y_j(t) = u(x_j,t) + n_j(t)$ $(j = 1,2,\ldots,\ \ell)$, where $n_j(t)$ $(j = 1,2,\ldots,\ \ell)$ denotes measurement noise.

One can relate the extracted modal coordinates to the actual measurements by[13]

$$q_M(t) = Gy(t) \tag{6}$$

where $y(t)$ is a vector of order ℓ containing the system measurements, and the entries of G are g_{rj} $(r = 1,2,\ldots,\ m;$ $j = 1,2,\ldots,\ \ell)$. It follows that the extracted modes are related to the actual modal coordinates by

$$q_M(t) = GH_M\boldsymbol{u}_M(t) + GH_R\boldsymbol{u}_R(t) + Gn(t) \tag{7}$$

where $H_M(i,j) = \phi_j(x_i)$ $(i = 1,2,\ldots,\ \ell;\ j = 1,2,\ldots,\ m)$ and $H_R(i,j) = \phi_{m+j}(x_i)$ $(i = 1,2,\ldots,\ \ell;\ j = 1,2,\ldots)$. The objective in modal filter design is to select the number and location of the sensors, the interpolation functions, and the number of monitored modes such that GH_M approaches an identity matrix and GH_R approaches a null matrix.

To investigate the minimum number of sensors required to implement modal filters, we observe from the preceding equation that G is of order $m*\ell$ and H_M is of order $\ell*m$. For GH_M to approach an identity matrix of order $m*m$, it has to have rank m. GH_M has rank m only if both the matrices G and H_M have rank m, which implies that $\ell \geq m$. The physical interpretation is that *the number of sensors must be at least as large as the number of monitored modes.* Note that this requirement on the minimum number of sensors does not guarantee accuracy of the modal filters.

When finite-element-type interpolation functions are used, the distributed domain is divided into p intervals D_j, and the displacement in each interval is approximated by $q_j(x,t)$ as

$$q_j(x,t) = \boldsymbol{L}^T(x)\boldsymbol{s}_j(t), \qquad j = 1,2,\ldots,\ p \tag{8}$$

in which $\boldsymbol{L}(x)$ denotes a set of local interpolation functions from the finite-element method, and $s_j(t)$ denote the set of measurements in the jth domain. Introducing Eq. (8) into Eq. (5) and performing the algebra, the extracted modal coordinates take the form[13]

$$q_r(t) = \sum_{j=1}^{p} \boldsymbol{h}_{jr}^T\boldsymbol{s}_j(t) \tag{9}$$

where

$$\boldsymbol{h}_{jr} = \int_{D_j} m(x)\phi_r(x)\boldsymbol{L}(x)\ \mathrm{d}x \tag{10}$$

Through assembly of $\boldsymbol{h}_{jr}$ for each sensor, the modal filter gain matrix G can be computed.

When Rayleigh-Ritz-type modal filters are used, the interpolation functions are global, and the interpolated profile has the form

$$q(x,t) = \sum_{r=1}^{\ell} \psi_r(x)\eta_r(t) \tag{11}$$

in which $\psi_r(x)$ are admissible functions. To determine the amplitudes $\eta_r(t)$, we express the system output as

$$y(t) = D\boldsymbol{\eta}(t) \tag{12}$$

where $D_{ij} = \psi_j(x_i)$ $(j = 1,2,\ldots;\ i = 1,2,\ldots,\ \ell)$ and $\boldsymbol{\eta}(t) = [\eta_1,\ \eta_2,\ldots]^T$. If the number of extracted modes is selected the same as the number of sensors, $\ell = m$, these coordinates can be computed by inverting the matrix constructed by the first ℓ columns of D (denoted by D_M) to yield

$$\boldsymbol{\eta}_M(t) = D_M^{-1}y(t) \tag{13}$$

in which $\boldsymbol{\eta}_M(t)$ denotes the first m entries of $\boldsymbol{\eta}(t)$. The term D_M^{-1} is the counterpart of G in Eq. (6). Substituting Eq. (13) into Eq. (11), the approximate deformation profile has the form

$$q(x,t) = \sum_{s=1}^{\ell} \psi_s(x) \sum_{j=1}^{\ell} [D_M^{-1}]_{sj}y_j(t) \tag{14}$$

and the entries of the modal filter gain matrix become

$$G_{rj} = \sum_{s=1}^{\ell} [D_M^{-1}]_{sj} \int m(x)\phi_r(x)\psi_s(x)\ \mathrm{d}x \tag{15}$$

A special situation is encountered when the admissible functions are selected as the eigenfunctions of the system. In this case, D_M becomes the same as H_M, and $\eta_M(t)=q_M(t)$, so that

$$q_M(t)=H_M^{-1}y(t) \tag{16}$$

where it is observed that Eq. (16) can be obtained without considering interpolation of the system output at all.

We next consider use of splines as interpolation functions. Here, one must determine which order spline functions to use. The answer is dependent on the number of boundary conditions one can specify.

For illustrative purposes, let us consider beam vibrations and use cubic splines as interpolation functions. As is well known, cubic splines yield functions with continuous first and second derivatives. We divide the beam length, taken as L, into k elements, each of length d_i $(i=1,2,\ldots,k)$ and denote the locations of the nodal points by x_i, such that $x_1=0$, $x_2=d_1$, $x_3=d_2+d_3,\ldots, x_{k+1}=L$. The number of sensors are related to k by the types of boundary conditions used. The approximate profile at the ith interval can be expressed as[19,20]

$$q_i(x,t)=A_i(x)M_i(t)+B_i(x)M_{i+1}(t)$$
$$+C_i(x)y_i(t)+D_i(x)y_{i+1}(t) \tag{17}$$

in which

$$A_i(x)=(1/6d_i)(x_{i+1}-x)^3-(d_i/6)(x_{i+1}-x),$$
$$B_i(x)=(1/6d_i)(x-x_i)^3-(d_i/6)(x-x_i)$$
$$C_i(x)=(1/d_i)(x_{i+1}-x),\ D_i(x)=(1/d_i)(x-x_i) \tag{18}$$

and $M_i(t)$ and $M_{i+1}(t)$ are the values of the second derivatives of the approximated profile at the nodal points, i.e., $M_i(t)=q''(x_i,t)$, $(i=1,2,\ldots,k+1)$. In standard spline interpolations, one knows y_i and finds M_i using the continuity relation for the first derivative. We will use the same approach, and set $q_i'(x_{i+1})=q_{i+1}'(x_{i+1})$, which yields for the intermediate nodes[19,20]

$$(d_i/6)M_i(t)+[(d_i+d_{i+1})/3]M_{i+1}(t)+(d_{i+1}/6)M_{i+2}(t)$$
$$=(y_i/d_i)-y_{i+1}[(1/d_i)+(1/d_{i+1})]+(y_{i+2}/d_{i+1})$$
$$i=1,2,\ldots,k-1 \tag{19}$$

The next step is to determine relations for the second derivatives at the end points. To accomplish this, we will consider the boundary conditions. If at $x=0$ there is a pinned or free end we can write

$$M_1(t)=0 \tag{20}$$

For a fixed end, or if the slope $q'(0,t)$ is prespecified, $q_1'(x,t)$ is evaluated at $x=0$, which yields[20]

$$q_1'(0,t)=-(d_1/3)M_1(t)-(d_1/6)M_2(t)-y_1(t)/d_1+y_2(t)/d_1 \tag{21}$$

In a similar fashion, one can find the appropriate relation for the end $x=L$. The $k-1$ relations of Eq. (19) and the two boundary terms are then assembled in matrix form as

$$PM(t)=Qy(t)+F(t) \tag{22}$$

where $M(t)=[M_1(t)\ M_2(t)\ \cdots\ M_{k+1}(t)]^T$, $[y(t)=[y_1(t)\ y_2(t)\ \cdots\ y_{k+1}(t)]^T$ and P and Q are tridiagonal matrices. They are symmetric for at least the second through $k-1$ rows and columns, depending on the boundary conditions. $F(t)$ is a vector containing expressions related to the slopes at the boundaries. For example, for a cantilever beam fixed at $x=0$, $F(t)=0$ and P and Q have the forms

$$P_{11}=d_1/3,\quad P_{k+1,k+1}=1,$$
$$P_{ii}=(d_i+d_{i+1})/3,\quad i=2,3,\ldots,k \tag{23a}$$
$$P_{i,i+1}=P_{i+1,i}=d_i/6,\qquad i=1,2,\ldots,k-1 \tag{23b}$$
$$P_{k,k+1}=d_k/6,\qquad P_{k+1,k}=0 \tag{23c}$$
$$Q_{11}=-1/d_1,\quad Q_{k+1,k+1}=0,$$
$$Q_{ii}=-(1/d_i+1/d_{i+1}),\quad i=2,3,\ldots,k \tag{23d}$$
$$Q_{i,i+1}=Q_{i+1,i}=1/d_i,\qquad i=1,2,\ldots,k-1 \tag{23e}$$
$$Q_{k,k+1}=1/d_k,\qquad Q_{k+1,k}=0 \tag{23f}$$

All other entries of P and Q are zero. One can then relate the values of $M(t)$ to the values of $y(t)$ by inverting Eq. (22), which yields

$$M(t)=P^{-1}Qy(t) \tag{24}$$

where it is observed that the matrices P and Q are not time dependent, so that the inversion has to be performed only once, before the modal filter extraction begins.

Next, we introduce the interpolated profile into the expansion theorem and carry out the algebra, which yields

$$q_r(t)=\sum_{i=1}^{k}\int_{D_i} m(x)\phi_r(x)q_i(x,t)\,\mathrm{d}x=A_r^TM(t)+B_r^Ty(t) \tag{25}$$

where

$$A_r=[a_{r1}\ \ b_{r1}+a_{r2}\ \ b_{r2}+a_{r3}\ \cdots\ b_{r,k-1}+a_{r,k}\ \ b_{r,k}]^T \tag{26a}$$
$$B_r=[c_{r1}\ \ d_{r1}+c_{r2}\ \ d_{r2}+c_{r3}\ \cdots\ d_{r,k-1}+c_{r,k}\ \ d_{r,k}]^T \tag{26b}$$

in which

$$a_{ri}=\int_{D_i} m(x)\phi_r(x)A_i(x)\,\mathrm{d}x,\quad b_{ri}=\int_{D_i} m(x)\phi_r(x)B_i(x)\,\mathrm{d}x \tag{27a}$$

$$c_{ri}=\int_{D_i} m(x)\phi_r(x)C_i(x)\,\mathrm{d}x,\quad d_{ri}=\int_{D_i} m(x)\phi_r(x)D_i(x)\,\mathrm{d}x \tag{27b}$$

Equation (25) can be expressed in the matrix form

$$q_M(t)=AM(t)+By(t) \tag{28}$$

where $A=[A_1\ A_2\ \cdots\ A_m]^T$ and $B=[B_1\ B_2\ \cdots\ B_m]^T$. When Eq. (28) is combined with Eq. (24), we obtain the relation between the extracted modal coordinates and system measurements as

$$q_M(t)=G_My(t)=[AP^{-1}Q+B]\ y(t) \tag{29}$$

For axial or torsional vibration problems, because there is only one boundary condition at each end, one cannot use cubic splines for interpolation. The spline functions suitable for such problems are quadratic splines,[20] which provide continuous profiles with continuous slopes. Their implementation is almost identical to the implementation of cubic splines and their description is not included here for the sake of brevity.

In the event of failure of a sensor, the performance of the modal coordinate extraction deteriorates. We will consider the following mode of failure. Denoting the actual measurement vector by $y_a(t)$, we relate it to the sensors output by

$$y_a(t) = S(t)y(t) \tag{30}$$

where $S(t)$ is a diagonal matrix denoting the level of failure.[9] Replacing $y(t)$ by $y_a(t)$ in Eq. (7), we obtain the relationship between the extracted modal coordinates and actual modal coordinates as[9]

$$q_M(t) = GS(t)H_M u_M(t) + GS(t)H_R u_R(t) + GS(t)n(t) \tag{31}$$

and the matrix $GS(t)H_M$ deviates from an identity matrix.

Next, we investigate the performance of modal filters in the presence of parameter uncertainties and eigensolutions obtained by spatial discretization. One must be concerned with two effects: 1) the deviation of the extracted modal coordinates from the actual modal coordinates and 2) the stability of the modal coordinate extraction. Consider model uncertainties in the form $m'(x)$ and L', where the primes denote that the mass and stiffness operators are erroneous and based on some postulated model. We develop a discretized model of order N using an approach such as the finite-element method or weighted residuals. Solution of the associated eigenvalue problem yields the computed eigenvalues Λ_r and eigenfunctions $\theta_r(x)$ ($r = 1,2,\ldots, N$), which obey the orthogonality relations $(\theta_r(x),\ m'(x)\theta_s(x)) = \delta_{rs}$, $[\theta_r(x),\ \theta_s(x)] = \Lambda_r\delta_{rs}$ $(r,s = 1,2,\ldots, N)$, where the square brackets denote the energy inner product associated with the postulated stiffness. The expansion theorem can be written as follows, where $v_r(t)$ are modal coordinates of the postulated discretized system:

$$u(x,t) = \sum_{r=1}^{N} v_r(t)\theta_r(x), \quad v_r(t) = (u(x,t),\ m'(x)\theta_r(x)) \tag{32}$$

When discrete sensors are used, the distributed profile can be approximated by Eq. (4), which, when substituted into the preceding equation, yields for the monitored postulated coordinates extracted from the system output $w(t)$ of order m (in the event of no failure)

$$w(t) = G'H_M u_M(t) + G'H_R u_R(t) + G'n(t) \tag{33}$$

or

$$w(t) = G'J_M v_M(t) + G'J_R v_R(t) + G'n(t) \tag{34}$$

where $G'_{rj} = [G(x,x_j),\ m(x)\theta_r(x)]$ ($r = 1,2,\ldots, m;\ j = 1,2,\ldots, \ell$); $J_M(i,j) = \theta_j(x_i)$ ($i = 1,2,\ldots, \ell;\ j = 1,2,\ldots, m$); and $J_R(i,j) = \theta_{m+j}(x_i)$ ($i = 1,2,\ldots, \ell;\ j = 1,2,\ldots, N-m$).

One designs the number and location of the sensors such that $G'J_M$ approaches an identity matrix and $G'J_R$ approaches a null matrix. However, the actual relationship between the extracted and actual modal coordinates is given by Eq. (33) so that the error incurred due to the model uncertainties and spatial discretization can be seen by how much $G'H_M$ deviates from an identity matrix.

III. Convergence Considerations in Modal Filters

Equation (5), which relates the extracted modal coordinates to the interpolated deformation profile $q(x,t)$, basically represents an orthogonal function expansion of $q(x,t)$. We can then make use of the convergence properties of orthogonal series expansions and of Fourier series to examine the convergence rates of modal filters. First, we summarize some results from Fourier series.

It is well known that given a function $f(x)$ defined between 0 and 2π, its Fourier series expansion is given by the function $g(x)$ with period 2π, such that

$$g(x) = \frac{a_0}{2} + \sum_{r=1}^{\infty}(a_r \sin rx + b_r \cos rx) \tag{35}$$

where the coefficients a_r and b_r can be found from[22]

$$a_r = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin rx \, dx \tag{36a}$$

$$b_r = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos rx \, dx \tag{36b}$$

It can be shown that if $f(x)$ is piecewise continuous and has bounded total variation,[22]

$$g(x) = 0.5[f(x^+) + f(x^-)] \tag{37}$$

What makes the expansion possible and convergent is the orthogonality of the expansion functions. It is of interest to determine the order of the coefficients a_r and b_r. This is related to how many times $f(x)$ can be differentiated. To demonstrate this, consider a Fourier sine series and integrate Eq. (36a) by parts, which yields [assuming that the values of $f(x), f'(x), f''(x),\ldots$ at $x = 0$ and $x = 2\pi$ permit the boundary terms to vanish]

$$a_r = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin rx \, dx = \frac{1}{\pi r}\int_0^{2\pi} f'(x) \cos rx \, dx$$
$$= \frac{-1}{\pi r^2}\int_0^{2\pi} f''(x) \sin rx \, dx = \cdots \tag{38}$$

The integration can be carried out until the kth derivative of $f(x)$ is piecewise continuous. It is shown in Ref. 22 that in such a case the coefficients a_r are of order $\mathcal{O}(1/r^{(k+1)})$. Also, there is a Gibbs effect in the expansion of $f^{(k)}(x)$.[22] The order of the coefficients a_r is larger if the boundary terms of the partial differentiation do not vanish, leading to a Gibbs effect in a lower order derivative of $f(x)$.

The preceding results are valid not only for a Fourier series expansion of a function, but also for any orthogonal series expansion of a function.[22] We propose to make use of Eqs. (35–38) to investigate convergence properties of the different types of interpolation functions in modal filtering. Using the expansion theorem, one can express Eq. (5) as

$$q_r(t) = \sum_{j=1}^{\ell}\sum_{s=1}^{\infty} g_{rj}\phi_s(x_j)u_s(t) = \sum_{s=1}^{\infty} e_{rs}u_s(t), \qquad r = 1,2,\ldots \tag{39}$$

where

$$e_{rs} = \Sigma g_{rj}\phi_s(x_j)$$

are the elements of GH in Eq. (7). As shown earlier, for modal filters to yield accurate results, e_{rs} must approach zero when $r \neq s$, which implies that g_{rj} must do so as well. Recalling the definition of g_{rj}, $g_{rj} = (G(x,x_j),\ m(x)\phi_r(x))$ ($j = 1,2,\ldots, \ell;\ r = 1,2,\ldots$), we observe that the issue at hand is the eigenfunction expansion of the interpolation functions.

When finite-element-type interpolation functions are used, if only displacement measurements are available, one can use linear or quadratic measurements that guarantee continuity at the external nodes. The output of a sensor, if at an external node, provides a profile of the displacement for the two elements connected to that node. For example, for linear elements, if we denote the element interpolation functions in the jth element by $L_{j1}(x)$ and $L_{j2}(x)$, $G(x,x_j)$ has the form

$$\begin{aligned} G(x,x_j) &= L_{j2}(x) && (j\text{th element}) \\ &= L_{j+1,1}(x) && (j+1\text{st element}) \\ &= 0 && (\text{elsewhere}) \end{aligned} \tag{40}$$

It follows that $G(x,x_j)$ has no continuous derivatives so that its eigenfunction expansion permits spatial differentiation only once. We then conclude that g_{rj} are order $\mathcal{O}(1/r^2)$. Note that if hierarchical finite elements are used, the quality of interpolation is improved inside the domain of the individual elements, but the continuity properties at the external nodes are not changed. The order of g_{rj} still remains $\mathcal{O}(1/r^2)$. This order can be improved to $\mathcal{O}(1/r^3)$ if slope measurements are taken or synthesized and hermite cubics are used as interpolation functions, which permits continuity of the slope profiles as well.

For the case of Ritz-type interpolations, one can select a variety of functions. Assume for the time being that the closed-form eigensolution is available. If these functions are used as the interpolation functions, we know that $e_{rs} = \delta_{rs}$, $(r,s = 1,2,..., m)$. Nothing, however, can be said about e_{rs} when either r or s is larger than m. It follows that g_{rj} are order $\mathcal{O}(1)$ for $r > m$ or $s > m$. This feature summarizes the advantages and disadvantages associated with using Rayleigh-Ritz-type interpolation functions. One obtains perfect accuracy with the monitored modes and totally ignores the residual dynamics. This situation can be improved by selecting the number of sensors used larger than the number of monitored modes.

When a closed-form eigensolution is not available and a solution is sought using spatial discretization, or if the interpolation functions used are admissible functions, convergence of modal filtering is dependent on the differentiability of the functions used. Since in most cases a finite-element approach is taken, we conclude that g_{rj} will be of order $\mathcal{O}(1/r^2)$ or $\mathcal{O}(1/r^3)$. This, of course, increases the values of e_{rs} for $r,s \leq m$. One has to go into specific cases to determine whether the increase in accuracy in the residual dynamics justifies the corresponding loss in the monitored modes.

For the case of cubic splines, continuity of the second derivative of the interpolated profile is guaranteed, leading to the conclusion that g_{rj} are of order $\mathcal{O}(1/r^4)$. For quadratic splines, g_{rj} are $\mathcal{O}(1/r^3)$. We observe then that as far as orders of convergence are concerned, using splines as interpolation functions is more desirable.

One final note about accuracy of modal filters. The convergence rates are very important quantities, but they do not depict the entire picture when determining the quality of modal filters. First of all, the number of sensors is of critical importance, a factor not observable from the orthogonal series expansion of $G(x,x_j)$. Second, while the convergence rate gives us an order, it does not quantify the actual coefficients. To assess the overall performance of the different types of interpolation functions, one must consider a quantitative analysis as well, which we carry out in the next section.

IV. Optimal Placement of Sensors

In this section, we quantitatively analyze the performance of modal filters and the accuracy of the modal coordinate extraction in the event of sensor malfunctions and when all components are operational and consider the placement of backup components.

We consider two models: bending vibration of a pinned-pinned beam and axial vibration of a tapered bar fixed at one end and free at the other. These two models are chosen because they both possess closed-form eigensolutions such that the accuracy of the results can be monitored. Also, the models have different characteristics. The tapered bar has closely spaced natural frequencies that increase with almost arithmetic progression, and the beam has frequencies that increase in geometric progression. The asymptotic behavior of the eigenvalues λ_r, as $r \to \infty$ is of order $\mathcal{O}(r^2)$ for the tapered bar and $\mathcal{O}(r^4)$ for the beam problem. In general, for a boundary-value problem of order $2p$, the asymptotic behavior of the eigenvalues is of order $\mathcal{O}(r^{2p})$ as $r \to \infty$.[23]

For the beam problem, the equation of motion and boundary conditions have the form

$$\mu(x)\ddot{u}(x,t) + \partial^2[EI(x)\partial^2 u(x,t)/\partial x^2]/\partial x^2 = f(x,t), \qquad 0 < x < L$$

$$u(0,t) = 0, \ EI(0)u''(0,t) = 0, \quad u(L,t) = 0, \ EI(L)u''(L,t) = 0 \tag{41}$$

where $u(x,t)$ is the transverse deformation at point x, $\mu(x)$ is the mass distribution in the form $\mu(x) = \mu$, $EI(x)$ is the stiffness as $EI(x) = EI$, and $f(x,t)$ is the external excitation, including controls. It is well known that this model admits a closed-form eigensolution

$$\phi_{1r}(x) = \sin r\pi x/L, \qquad r = 1,2,... \tag{42}$$

where the subscript 1 denotes that it is the first model used.

For the axial vibration problem, the equation of motion is

$$\rho(x)\ddot{u}(x,t) = \partial[EA(x)\partial u(x,t)/\partial x]/\partial x + f(x,t), \ 0 < x < L \tag{43}$$

$$u(0,t) = 0, \ EA(L)u'(L,t) = 0$$

where $u(x,t)$ is the axial deformation at point x, $\rho(x)$ is the mass distribution in the form $\rho(x) = 2\rho(1 - x/L)$, $EA(x)$ is the stiffness as $EA(x) = 2EA(1 - x/L)$, and $f(x,t)$ is the external excitation, including controls. The eigensolution can be shown to be in the form

$$\phi_{2r}(x) = J_0[\beta_r L(1 - x/L)], \qquad r = 1,2,... \tag{44}$$

where J_0 is the Bessel function of order zero, and β_r are roots of the characteristic equation $J_0(\beta_r L) = 0$ from which the natural frequencies are obtained. For both the bar and beam, the length is taken as $L = 10$. Also, $\mu = \rho = 1$, $EI = 1$, and $EA = 1$.

We first consider optimization of the sensor locations when all sensors are operational. To accomplish this, we define an

Table 1 Optimal sensor locations for pinned-pinned beam, failure not considered

	Sensor Locations						
Initial	1.66667	3.33333	5.00000	6.66667	8.33333	J	(initial J)
Finite-Element	1.67713	3.38286	5.00005	6.61749	8.32293	0.71744	(0.72693)
Rayleigh-Ritz	1.65699	3.35362	5.00000	6.64638	8.34301	0.09697	(0.10025)
Cubic Splines	1.65597	3.35366	5.00002	6.64634	8.34403	0.08698	(0.08970)

Table 2 Optimal sensor locations for tapered bar, failure not considered

	Sensor Locations (first through fifth, sixth at 10.0)						
Initial	1.66667	3.33333	5.00000	6.66667	8.33333	J	(initial J)
Finite-Element	1.60193	3.35253	4.96594	6.47589	7.88085	0.58216	(0.70839)
Rayleigh-Ritz	1.60471	3.20996	4.81592	6.42508	8.04769	0.00002	(0.10550)
Quadr. Splines	1.59247	3.20006	4.80338	6.40286	8.00949	0.01030	(0.25363)

Table 3 GH_M and GH_R matrices for pinned-pinned beam, finite-element-type modal filters

$$GH_M = \begin{bmatrix} 0.998576 & 0.000003 & 0.005162 & 0.000017 \\ 0.000001 & 0.983867 & 0.000020 & 0.210953 \\ 0.000017 & -0.000001 & 1.032049 & 0.000013 \\ 0.000001 & 0.035120 & -0.000020 & 0.800622 \end{bmatrix}$$

$$GH_R = \begin{bmatrix} 0.301068 & 0.000069 & -0.279459 & -0.000107 \\ 0.000023 & 0.026945 & 0.000048 & 0.267547 \\ 0.006017 & -0.000011 & -0.030513 & -0.000107 \\ -0.000008 & -0.050268 & -0.000174 & -0.766629 \end{bmatrix}$$

Table 4 GH_M and GH_R matrices for pinned-pinned beam, Rayleigh-Ritz-type modal filters

$$GH_M = \begin{bmatrix} 1.000000 & 0.000000 & 0.000000 & 0.000000 \\ 0.000000 & 1.000000 & 0.000000 & 0.000000 \\ 0.000000 & 0.000000 & 1.000000 & 0.000000 \\ 0.000000 & 0.000000 & 0.000000 & 1.000000 \end{bmatrix}$$

$$GH_R = \begin{bmatrix} 0.000000 & 0.000004 & 0.032744 & -0.000002 \\ 0.000000 & 0.032744 & 0.000005 & -0.011858 \\ 0.000000 & 0.000002 & 0.021262 & 0.000002 \\ 0.000000 & -0.011482 & -0.000001 & -0.967124 \end{bmatrix}$$

Table 5 GH_M and GH_R matrices for pinned-pinned beam, cubic splines

$$GH_M = \begin{bmatrix} 0.999889 & 0.000000 & -0.000584 & 0.000002 \\ 0.000000 & 0.997756 & 0.000000 & -0.000337 \\ -0.000007 & 0.000000 & 0.985518 & -0.000002 \\ 0.000000 & -0.000021 & 0.000000 & 0.939055 \end{bmatrix}$$

$$GH_R = \begin{bmatrix} 0.006552 & -0.000014 & 0.027274 & 0.000027 \\ -0.000001 & 0.033839 & 0.000001 & -0.010428 \\ 0.007790 & 0.000014 & 0.015275 & -0.000025 \\ 0.000003 & -0.009790 & -0.000001 & -0.907108 \end{bmatrix}$$

objective function that provides a measure of the accuracy of the modal coordinate extraction. As stated earlier, the objective is to have the matrix GH_M approach an identity matrix and to have GH_R approach a null matrix. From Eq. (8), the error in modal filtering can be defined as

$$u_M(t) - q_M(t) = [I - GH_M]u_M(t) - GH_R u_R(t) - Gn(t) \tag{45}$$

In addition, because the lower modes contribute more to the system output, they need to be extracted with better accuracy than the higher modes. Another way of stating this is that, because the higher modes have lower amplitudes, they can tolerate more errors in their extraction. We then define the performance functional that has to be minimized as $J = J(x_1, x_2, \dots, x_\ell)$, where

$$J = \sum_{i=1}^{m}\left[\sum_{j=1}^{m}(GH_M - I)_{ij}^2/\omega_i\omega_j + \sum_{j=1}^{n-m}(GH_R)_{ij}^2/\omega_i\omega_{m+j}\right]$$

$$\text{s.t. } 0 < x_i \le L, \qquad i = 1,2,\dots,\ell;$$

$$x_{i-1} < x_i < x_{i+1}, \qquad i = 2,3,\dots,\ell-1 \tag{46}$$

in which the elements of GH_M and GH_R are weighed depending on the natural frequencies of the modes they affect. The following parameters were used for the optimization. For the pinned-pinned beam, the number of monitored modes was taken as $m = 4$, the number of residual modes was taken as $n - m = 4$, and the number of sensors was selected as $\ell = 5$. For the axial vibration problem, the parameters selected were $m = 4$, $n = 8$, and $\ell = 6$. For the tapered bar, one of the

Table 6 Effect of failure of a sensor by 50% on the objective function J_k, pinned-pinned beam (FE: finite-element, RR: Rayleigh-Ritz, CS: cubic splines)

Faulty sensor		FE	RR	CS
J_k	$k = 1$	1.910348	1.131945	1.123407
	2	2.692348	2.998509	2.983163
	3	5.983435	3.812228	3.800102
	4	2.692348	2.998509	2.983163
	5	1.910346	1.131945	1.123407

Table 7 Effect of failure of a sensor by 50% on the objective function J_k, tapered bar (QS: quadratic splines)

Faulty sensor		FE	RR	QS
J_k	$k = 1$	1.027193	0.499249	0.621307
	2	1.344897	0.725276	0.853883
	3	1.389944	0.895112	1.042774
	4	1.573913	0.984729	1.122892
	5	1.295084	0.791381	0.921378
	6	0.825442	0.241735	0.382878

sensors was placed at the tip of the bar, i.e., $x_\ell = L$, and its location was not varied throughout the optimization.

The golden section search method,[24] combined with a first-order gradient technique, is used for finding the minimum values of J. In this approach, first the direction of steepest descent (gradient) is obtained by numerical differentiation. Then, using the golden section technique, the minimum point in the direction of the gradient is obtained. The procedure is repeated until the norm of the gradient or the difference in objective functions between adjacent steps reach a threshold value. Also, even though Eq. (46) defines a constrained problem, the optimization procedure used was an unconstrained one. This is because we expected the optimal sensor distributions to be even and selected our initial distributions relatively evenly.

Tables 1 and 2 compare the results of the optimization for the three sets of interpolation functions for the two models considered. The Rayleigh-Ritz-type modal filters are implemented by using the eigenfunctions of the system. For the tapered bar, when splines are used, quadratic splines are considered. For beam vibrations, cubic splines are used. We observe that the three interpolation approaches work with comparable accuracy, with Rayleigh-Ritz-type filters giving slightly better results for the tapered bar. The optimal sensor locations obtained from all three analyses are very similar to each other, which is to be expected. Comparing the initial and final values of objective function, we observe that there is not too much change, so that a relatively even spreading of the sensors yields satisfactory results. For the tapered bar, the optimal locations tend to move towards the fixed end, which has a larger mass distribution, an expected behavior.

Tables 3-5 show the GH matrices for the optimal sensor locations for the pinned-pinned beam. Naturally, GH_M should approach an identity matrix and GH_R should approach a null matrix. As can be seen, the accuracy of the modal filter extraction is comparable, with finite-element-type filters yielding slightly less accurate results. Rayleigh-Ritz-type filters give a GH_M matrix that is an identity matrix, but these filters result in the GH_R matrix having higher values, a consequence of their not considering the modes that are not modeled. Note that the first column of GH_R is identically zero because, with five sensors and four monitored modes, Rayleigh-Ritz-type modal filters can eliminate the contribution of the first residual mode to the monitored modes exactly.

Next we consider sensor malfunctions. The objective is to place the sensors such that in the event of sensor failure, the

Table 8 Optimal sensor locations for pinned-pinned beam to accommodate failures

	Sensor Locations						
Initial	1.66667	3.33333	5.00000	6.66667	8.33333	J	(initial J*)
Finite-Element	1.75782	3.48380	4.99973	6.51565	8.24197	2.96624	3.03765
Rayleigh-Ritz	1.75295	3.46418	4.99997	6.53579	8.24700	2.36980	2.41463
Cubic Splines	1.76159	3.48185	4.99933	6.51695	8.23796	2.35501	2.40265

Table 9 Optimal sensor locations for tapered bar to accommodate failures

	Sensor Locations (first through fifth, sixth at x = 10)						
Initial	1.66667	3.33333	5.00000	6.66667	8.33333	J*	(initial J*)
Finite-Element	1.61250	3.37376	4.97177	6.47798	7.88159	1.34530	1.49130
Rayleigh-Ritz	1.61679	3.23173	4.82895	6.42578	8.03680	0.70829	0.82750
Quadr. Splines	1.59224	3.19788	4.79919	6.39178	7.99471	0.69422	0.98902

Table 10 Objective function when faulty sensor is replaced by a backup, one backup sensor, finite-element interpolation

Faulty	Backup locations			
sensor	1	2	3	4
1	1.451057	3.833609	4.084225	3.902011
2	2.543774	2.547080	6.140441	5.976546
3	6.516946	2.525715	2.524318	6.517659
4	5.975988	6.140352	2.546187	2.544318
5	3.900320	4.083303	3.832231	1.451133
Total	20.388085	19.130058	19.127402	20.391667

Table 11 Objective function when faulty sensor is replaced by a backup, one backup sensor, Rayleigh-Ritz interpolation

Faulty	Backup locations			
sensor	1	2	3	4
1	1.813261	29.922001	102.461696	172.683969
2	1.668439	2.526517	24.510521	53.698627
3	14.658159	2.351099	2.351079	14.658182
4	53.698223	24.510398	2.526484	1.668446
5	172.680824	102.460315	29.921537	1.813252
Total	244.518907	161.770330	161.771317	244.522475

Table 12 Objective function when faulty sensor is replaced by a backup, one backup sensor, cubic splines

Faulty	Backup locations			
sensor	1	2	3	4
1	0.858790	3.039593	3.342801	3.161065
2	1.477237	1.657844	4.582862	4.827636
3	5.044105	1.819949	1.820020	5.044106
4	4.827414	4.582699	1.657815	1.477095
5	3.161089	3.342811	3.039557	0.858759
Total	15.368635	14.442897	14.443055	15.368660

loss of accuracy in the modal coordinate extraction will be minimized. We investigate the effects of failure by comparing the value of the objective function in the presence of faulty sensors. The performance functional is modified to show the effects of failure as

$$J_k = \sum_{i=1}^{m}\left[\sum_{j=1}^{m}(GS_kH_M - I)_{ij}^2/\omega_i\omega_j + \sum_{j=1}^{n-m}(GS_kH_R)_{ij}^2/\omega_i\omega_{m+j}\right], \qquad k = 1,2,\ldots, \ell \qquad (47)$$

in which J_k denotes the value of the objective function calculated with the sensors of initial locations as given in Tables 1 and 2, and S_k is a diagonal matrix with its kth diagonal element denoting the level of failure in the kth sensor. Tables 6 and 7 give the values of J_k for a level of failure of 0.5 for each sensor. Comparing these results with Tables 1 and 2, we observe that for both models failure of a sensor adversely affects the value of the objective function and that failure of a sensor in or around the middle of the domain affects the accuracy of modal coordinate extraction more.

We consider next the optimization of the sensors' locations to accommodate failures. We consider the objective function

$$J^* = \sum_{k=1}^{\ell} J_k w(k) \qquad (48)$$

where J_k is defined in Eq. (47) where $w(k)$ is a weighting function such that $\Sigma w(k) = 1$. In each case of failure, the faulty sensor is assumed to read only 50% of the actual displacement, and the probability of failure for each sensor is assumed equal, leading to $w(k) = 1/\ell$. The optimization was conducted using the same approach described earlier. The results are shown in Tables 8 and 9 for the two models used. From these tables, we can draw two conclusions. First, the optimal locations of the sensors that minimize the effects of malfunctions are very close to the optimal locations obtained when failure is not considered—especially for the tapered bar. Second, Rayleigh-Ritz-type filters and splines do a comparable job, which is better than finite-element filters, with splines leading to slightly lower values of the objective function J^*.

We next investigate the number and placement of backup components. When a control system component is diagnosed as faulty, one has a number of options and considerations. If the level of failure is low, one may get away with doing nothing. If the sensor has to be replaced, two problems emerge: 1) which one of the backups to use and 2) how to reconfigure the control system. Intuitively, one can expect that providing more backups increases the reliability of the control design. However, with the large number of sensors expected to be used in future structures, it will not be feasible to provide a backup for every operational component.

We consider first the case of a single backup sensor. The candidate locations are selected as follows. Denoting the operational sensors' locations by $x_1, x_2, \ldots, x_\ell$, the candidate locations are chosen as $z_i = (x_i + x_{i+1})/2$ $(i = 1,2,\ldots, \ell - 1)$. Tables 10–12 display the value of the objective function J for the candidate backup locations in the event of failure for the pinned-pinned beam, where the sensor locations are taken as those in Table 1. The ji element of Table 1, denoted by J_{ji}, corresponds to the value of J [Eq. (46)] when the jth sensor is faulty, it is removed from operation and the backup sensor in location i is brought on-line. As expected, the results are better if the backup component is located close to the faulty sensor. We also observe that spline functions yield better results, and the results become especially undesirable for the case of Rayleigh-Ritz-type filters, which demonstrate themselves to be

Table 13 Objective function when faulty sensor is replaced by a backup, one backup sensor, finite-element interpolation

Faulty sensor	Backup locations											
	1	2	3	4	5	6	7	8	9	10	11	12
1	0.271	0.277	0.310	0.322	0.305	0.359	0.316	0.328	0.297	0.329	0.344	0.349
2	0.629	0.320	0.336	0.630	0.532	0.553	0.520	0.521	0.504	0.539	0.557	0.563
3	0.823	0.805	0.318	0.372	0.803	0.862	0.841	0.862	0.826	0.860	0.873	0.877
4	1.045	1.044	1.092	0.506	0.488	1.053	0.979	0.954	0.957	0.988	1.014	1.023
5	1.141	1.132	1.069	0.991	0.326	0.534	0.906	0.940	0.980	1.020	1.043	1.053
6	1.013	1.007	0.983	0.985	1.063	0.396	0.404	1.024	0.918	0.937	0.950	0.952
7	0.747	0.741	0.719	0.688	0.608	0.544	0.277	0.328	0.658	0.712	0.757	0.770
8	0.940	0.934	0.907	0.884	0.839	0.889	0.969	0.505	0.464	0.977	0.933	0.922
9	0.977	0.972	0.955	0.938	0.930	0.924	0.884	0.852	0.284	0.390	0.903	0.911
10	0.446	0.440	0.418	0.395	0.363	0.387	0.391	0.430	0.516	0.296	0.296	0.532
11	0.398	0.394	0.375	0.359	0.351	0.353	0.354	0.358	0.313	0.328	0.277	0.289
Total	8.429	8.067	7.482	7.069	6.609	6.853	6.841	7.103	6.717	7.376	7.946	8.242

Table 14 Objective function when faulty sensor is replaced by a backup, one backup sensor, Rayleigh-Ritz interpolation

Faulty sensor	Backup locations											
	1	2	3	4	5	6	7	8	9	10	11	12
1	0.041	0.132	0.916	2.556	8.048	22.326	34.197	40.482	45.771	76.810	130.861	197.076
2	0.703	0.094	0.153	1.063	4.653	14.992	24.261	29.855	34.835	59.644	102.455	154.945
3	3.563	1.441	0.124	0.216	2.233	9.190	16.291	21.373	25.813	44.785	77.715	118.466
4	10.431	5.720	1.802	0.223	0.232	3.461	8.103	12.490	16.836	31.209	55.723	86.225
5	19.682	11.979	4.858	1.405	0.155	0.534	2.584	5.393	8.282	16.004	29.371	46.410
6	24.902	16.122	7.649	3.313	1.535	0.181	0.119	1.090	2.661	6.328	12.609	20.762
7	28.055	18.618	9.185	4.186	2.419	0.896	0.105	0.135	0.770	2.126	4.564	7.917
8	64.064	43.439	22.653	11.771	8.455	4.967	1.982	0.202	0.137	1.207	3.450	6.758
9	163.123	111.903	59.302	31.176	23.664	16.462	8.550	2.205	0.130	0.184	1.327	3.587
10	115.598	80.046	43.458	24.128	19.708	15.187	9.074	3.289	0.684	0.075	0.064	0.492
11	393.037	272.652	147.897	81.273	66.585	52.980	32.912	12.978	3.646	1.202	0.207	0.064
Total	823.199	562.147	297.996	161.310	137.687	141.176	138.178	129.492	139.566	239.574	418.345	642.701

Table 15 Objective function when faulty sensor is replaced by a backup, one backup sensor, cubic spline interpolation

Faulty sensor	Backup locations											
	1	2	3	4	5	6	7	8	9	10	11	12
1	0.032	0.066	0.125	0.140	0.133	0.147	0.130	0.140	0.132	0.138	0.138	0.134
2	0.263	0.074	0.074	0.260	0.300	0.310	0.308	0.307	0.307	0.306	0.307	0.304
3	0.559	0.507	0.107	0.140	0.533	0.589	0.552	0.568	0.552	0.564	0.564	0.559
4	0.728	0.741	0.677	0.172	0.158	0.684	0.726	0.727	0.732	0.726	0.727	0.724
5	0.759	0.760	0.763	0.662	0.113	0.319	0.691	0.752	0.753	0.758	0.756	0.754
6	0.644	0.649	0.642	0.653	0.591	0.137	0.088	0.514	0.614	0.645	0.649	0.640
7	0.450	0.451	0.454	0.453	0.437	0.369	0.082	0.102	0.416	0.453	0.452	0.454
8	0.649	0.650	0.647	0.659	0.641	0.662	0.607	0.168	0.131	0.596	0.656	0.645
9	0.651	0.654	0.653	0.655	0.663	0.656	0.646	0.571	0.076	0.161	0.585	0.659
10	0.230	0.232	0.230	0.237	0.220	0.243	0.223	0.234	0.207	0.046	0.053	0.218
11	0.176	0.178	0.176	0.181	0.177	0.183	0.175	0.178	0.157	0.140	0.069	0.048
Total	5.140	4.963	4.546	4.214	3.965	4.299	4.226	4.260	4.078	4.534	4.957	5.137

Table 16 Minimum values of J_T, using objective function given by Eq. (46)

No. of backups	FE	RR	CS
1	6.608	129.492	3.966
2	5.368	13.491	2.814
3	4.523	7.274	1.975
4	3.870	4.423	1.318
5	3.767	2.342	1.076
6	3.731	1.467	0.983
7	3.723	1.324	0.955
8	3.717	1.265	0.955
9	3.717	1.256	0.955
10	3.717	1.256	0.955
11	3.717	1.256	0.955
12	3.717	1.256	0.955

Table 17 Minimum values of J_T, using objective function given by Eq. (53)

No. of backups	FE	RR	CS
1	46.606	23.901	19.063
2	38.475	9.325	15.241
3	31.031	6.955	12.340
4	26.479	5.173	9.539
5	23.718	3.841	7.877
6	22.198	3.358	7.086
7	21.922	3.268	6.847
8	21.647	3.191	6.742
9	21.647	3.191	6.724
10	21.647	3.191	6.724
11	21.647	3.191	6.724
12	21.647	3.191	6.724

extremely sensitive to the sensor locations. This sensitivity arises because of the inability of these filters to accommodate the residual modes. Note that for Rayleigh-Ritz-type filters the value of J is due to solely the second term in Eq. (46).

To identify the optimal location of the backup component, values of the objective functions for each case of failure are summed for the candidate locations using the relationship

$$J_{Ti} = \sum_{j=1}^{\ell} J_{ji}, \qquad i = 1,2,\ldots, \ell - 1 \tag{49}$$

and given in the bottom of Tables 10–12. We observe that for the cases of splines and finite-element-type interpolations, the location of the backups is not very critical, with placement around the middle of the domain giving slightly better results.

If the results of Tables 10–12 are compared with Table 6, we observe that replacing a faulty sensor with a backup far away from it results in a very high value of the objective function. This implies that if the level of failure is low, and if the location of the backup is too far away from the faulty component, it may be wiser not to bring the backup on-line. This problem disappears gradually as the number of sensors and backups are increased because a large number of sensors leads to more evenness. To investigate this, we increase the number of sensors to $\ell = 11$ and consider 12 candidate backup locations, at $z_i = (x_i + x_{i+1})/2$ $(i = 0,1,\ldots, \ell)$, with $x_0 = 0$ and $x_{\ell+1} = L$. The number of monitored modes is selected as $m = 7$, and we include 7 residual modes in the mathematical model. Due to hardware or geometry restrictions, the sensors cannot always be placed at their optimally calculated locations. The operational sensor locations are therefore selected using the relation $x_i = i\ell/(\ell + 1) + R$ $(i = 1,2,\ldots, \ell)$, where R is a random variable with a uniform distribution of $[-0.1, 0.1]$. The results for J_{ji} are given in Tables 13–15. As expected, values of the objective function become lower—especially for Rayleigh-Ritz-type interpolations.

Next we examine the number of backup sensors required for a reliable operation. We denote by n_b the number of backup components. It follows that there are $P = (\ell+1)!/(n_b)!(\ell+1-n_b)!$ possible backup configurations. We denote the locations of the backups by $z_{1k}, z_{2k}, \ldots, z_{n_b k}$ $(k = 1,2,\ldots, P)$ for the kth configuration. Considering failure of the jth sensor, we define the value of the objective function when the jth sensor is replaced by the ith backup as

$$J_{ji}^{(k)} = J(x_1, x_2, \ldots, x_{j-1}, \quad x_{j+1}, \ldots, x_\ell, z_{ik})$$
$$i = 1,2,\ldots, n_b, \qquad k = 1,2,\ldots, P \tag{50}$$

and for each set k and failure of sensor j, the minimum value of $J_{ji}^{(k)}$ identifies which backup to use when the jth sensor fails, so that

$$J_j^{(k)} = \min\left(J_{j1}^{(k)}, J_{j2}^{(k)}, \ldots, J_{jn_b}^{(k)}\right)$$
$$j = 1,2,\ldots, \ell, \qquad k = 1,2,\ldots, P \tag{51}$$

Then, for the entire set of sensor failures for the kth configuration we define

$$J_T^{(k)} = \sum_{j=1}^{\ell} J_j^{(k)}, \qquad k = 1,2,\ldots, P \tag{52}$$

which denotes the minimum value of the objective function for the kth backup configuration. To find the configuration that yields the smallest value of the objective function we select the minimum value of $J_T^{(k)}$ $(k = 1,2,\ldots, P)$ as $J_{T\min} = \min(J_T^{(1)}, J_T^{(2)}, \ldots, J_T^{(P)})$.

Table 16 displays the values of $J_{T\min}$ for varying numbers of backup sensors and the three interpolation functions. We observe that, for each interpolation, having five backups is sufficient, and increasing the number of sensors beyond that does not result in too much additional reliability. We also observe that splines do a better job. It should be noted that in the combination of backup sensor locations that gave the minimum values of $J_{T\min}$, the locations were all evenly distributed, but they were not the same locations for the different interpolation functions.

The reason splines perform better than Rayleigh-Ritz filters for this case is related to the contribution from the residual dynamics. One can then raise the question as to how the results of the optimization would vary if the objective function were changed to one which weighs the residual dynamics even less. To investigate this, we define the following objective function

$$J = \sum_{i=1}^{m}\left[\sum_{j=1}^{m} |GH_M - I|_{ij}/(\omega_i\omega_j)^{3/2} + \sum_{j=1}^{n-m} |GH_R|_{ij}/(\omega_i\omega_{m+j})^{3/2}\right] \tag{53}$$

Table 17 shows the values of $J_{T\min}$ for the preceding objective function. We observe that the results for Rayleigh-Ritz filters have improved substantially and are better than the results for cubic splines. However, the counterpart of Tables 13–15 for this case (which are not given here for the sake of brevity) still have the off-diagonal elements that are very high for Rayleigh-Ritz filters.

Tables 16 and 17 indicate that the results of the optimization and choice of the interpolation function are very much dependent on the amplitudes involved in the residual dynamics. If the system is highly damped, so that the modes that are not monitored have negligible amplitudes, the Rayleigh-Ritz-type filters do a better job. If, on the other hand, the residual dynamics has some significance and failure of more than one sensor is considered, or if the number of backups is limited, splines do an overall better job. The decision as to which type interpolation function to use should be taken considering the preceding factors as well as general convergence characteristics.

V. Conclusions

The optimal placement of sensors and of their backups is considered for distributed-parameter systems for cases when the sensors' output is used to extract modal coordinates from the system output. Three different interpolation functions are compared to implement modal filters. Of these, spline functions are found to be more desirable—especially in the cases of partial failures and when a backup is to be brought on-line. The results indicate that a relatively even placement of the sensors gives satisfactory results, with an uneven distribution leading to inaccuracies in the modal coordinate extraction. The optimal locations of the sensors to accommodate failures are very close to the optimal locations found when all sensors are operational. Also, accuracy of the modal filter extraction is relatively insensitive to the placement of the backup components as long as the backups are placed somewhat evenly. The number and location of the backups should be chosen such that, after a faulty sensor is replaced by a backup, the resulting sensor distribution will also be relatively even.

Acknowledgment

The research reported herein was supported by National Science Foundation Grant MEA-85-01877.

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